Relative Quaternion State Transition Relation

Ronald A. Mayo*
TRW Defense and Space Systems Group, Redondo Beach, Calif.

The attitude of a maneuvering spacecraft relative to a desired noninertial reference is compactly represented in the quaternion format by the relative quaternion. The popular technique for bootstrapping the relative quaternion relies on the state transition matrix for the quaternion strapdown equations of motion wherein the rates are estimates of spacecraft rates relative to the desired reference written in body coordinates. Even with a perfect three-axis gyro pack, whose signals are noiseless and always proportional to spacecraft inertial rates, the mere fact that the transformation from reference to body coordinates is not exact causes the relative quaternion estimate by the popular technique to diverge from the truth. It is shown that the rate of divergence from the truth is a function of the post-update attitude error, the maneuver rate, and the gyro sample frequency. An alternate form of the state transition matrix is derived which is invariant under all transformations from reference to body coordinates. With perfect gyros and for a spacecraft spinning at a constant rate, the error in the relative quaternion estimate, using the invariant state transition matrix, remains bounded to the postupdate attitude error.

Introduction

MANEUVERING, spinning spacecraft of the future will require an onboard attitude reference system nominally consisting of a three-axis strapdown gyro pack, a stellar-inertial and/or an Earth reference sensor for attitude updates, and an onboard computer for near or complete autonomous performance. Spacecraft inertial rate estimates based on imperfect gyro data are used to estimate the spacecraft inertial attitude in the quaternion format. Command signals to the control actuators are synthesized from error signals which, in part, depend on the spacecraft inertial rate estimates and the attiude or quaternion estimate. When maneuvering from one attitude to another or tracking a rotating Earth reference, the spacecraft attitude can be slaved to a desired noninertial reference that moves relative to inertial (ECI) space in a precomputed fashion. The command signals to the control actuators are then synthesized using relative rates and relative attitude errors. For this application, the relative quaternion describing the attitude of a body fixed orthogonal triad B relative to the noninertial desired reference R can be bootstrapped forward in time using sampled gyro data and appropriate onboard computer software.

For a spacecraft that is required to spin about its z axis while this axis points to nadir, the desired reference becomes a spinning Earth tracking reference. A candidate attitude reference system (ARS) consists of a three-axis gyro pack, slit-type star mappers, Earth sensors, and an onboard computer. The motion of R space, the desired reference, is not known but can be estimated based on the estimated orbital motion. The spacecraft is first maneuvered to Earth acquisition and tracks the nadir using the Earth sensors and gyros. Refined attitude estimates are obtained by spinning the spacecraft about its z axis. The onboard desired reference is synthesized using initial attitude and ephemeris estimates and becomes the estimate $\hat{R}$. $\hat{R}$ is updated infrequently and in an adaptive fashion using the Earth sensor data to indicate the need for a reference update. The reference $\hat{R}$ is updated using ephemeris and star mapper data. The distinction between R and its estimate $\hat{R}$ will not be treated here. What is important is the fact that $\hat{R}$ is known very accurately relative to inertial space and therefore star sighting can be used to update the relative quaternion. This paper is restricted to the analytical behavior of the relative quaternion between updates.

The relative quaternion,

$$q=\chi+i\xi+j\eta+k\zeta \tag{1}$$

is a function of the Euler symmetric parameters χ, ξ, η, and ζ which satisfy the differential equation

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \\ \dot{\chi} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \Delta\omega_z & -\Delta\omega_y & \Delta\omega_x \\ -\Delta\omega_z & 0 & \Delta\omega_x & \Delta\omega_y \\ \Delta\omega_y & -\Delta\omega_x & 0 & \Delta\omega_z \\ -\Delta\omega_x & -\Delta\omega_y & -\Delta\omega_z & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \chi \end{bmatrix} \tag{2}$$

where $(\Delta\omega_x,\ \Delta\omega_y,\ \Delta\omega_z)^T$ is the rate of B space relative to R space written in B-space coordinates. The $\boldsymbol{i,j,k}$ in the context of quaternions are hyperimaginary numbers satisfying very specific conditions.[1]

A popular technique for bootstrapping q forward in time is based on Eq. (2). The quaternion q is propagated forward in time using the well-known closed-form solution of the quaternion strapdown equation

$$q(t+\Delta t) = \{\cos(\|\Delta\omega\|\Delta t/2)I + [\sin(\|\Delta\omega\|\Delta t/2)/\|\Delta\omega\|]\Delta\Omega\}q(t) \tag{3}$$

where q in this context is the four-tuple $(\xi,\ \eta,\ \zeta,\ \chi)^T$ and $\Delta\Omega$ is the 4×4 matrix of relative rates in Eq. (2). The relative rates $\Delta\omega_x$, $\Delta\omega_y$, and $\Delta\omega_z$ are defined by

$$\begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\omega_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} - A_{B/R} \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \tag{4}$$

where

$(\omega_x,\omega_y,\omega_z)^T$ = inertial rate of B space in B space coordinates
$(\Omega_x,\Omega_y,\Omega_z)^T$ = inertial rate of R space in R space coordinates
$A_{B/R}$ = transformation from R space to B space

In practice, the onboard attitude reference system tracks estimates of these variables. As the estimate of $q(t)$, $\hat{q}(t)$,

Received Aug. 1, 1977; revision received April 24, 1978.

Index categories: Spacecraft Dynamics and Control; Spacecraft Simulation; Analytical and Numerical Methods.

*Member of the Technical Staff, Control Systems Engineering Dept.

drifts from the truth, the transformation $A_{\hat{B}/R}$ which is a function of $\hat{q}(t)$ introduces an error in the computation of the relative rates and, thus, in the propagation of the attitude estimate forward in time. If one assumes perfect gyros, it is found that when the spacecraft is maneuvering at a constant rate, the rate of divergence of $\hat{q}(t)$ from the truth is a function of the post-update attitude error, the maneuver rate, and the sample frequency. It will be shown herein that this error can be eliminated by using the state transition matrix based on the alternate form of the relative quaternion strapdown equation

$$\dot{q}(t) = \tfrac{1}{2}\Omega q(t) \tag{5}$$

where

$$\Omega = \begin{bmatrix} 0 & (\omega_z+\Omega_z) & -(\omega_y+\Omega_y) & (\omega_x-\Omega_x) \\ -(\omega_z+\Omega_z) & 0 & (\omega_x+\Omega_x) & (\omega_y-\Omega_y) \\ (\omega_y+\Omega_y) & -(\omega_x+\Omega_x) & 0 & (\omega_z-\Omega_z) \\ -(\omega_x-\Omega_x) & -(\omega_y-\Omega_y) & -(\omega_z-\Omega_z) & 0 \end{bmatrix} \tag{6}$$

The strapdown equation expressed in Eqs. (5) and (6) can be derived from Eqs. (2) and (4) using

$$YZ = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & A_{B/R} & \\ 0 & & & \end{array}\right] \tag{7}$$

where

$$Y = \begin{bmatrix} \chi & \xi & \eta & \zeta \\ -\xi & \chi & \zeta & -\eta \\ -\eta & -\zeta & \chi & \xi \\ -\zeta & \eta & -\xi & \chi \end{bmatrix} \tag{8}$$

and

$$Z = \begin{bmatrix} \chi & -\xi & -\eta & -\zeta \\ \xi & \chi & \zeta & -\eta \\ \eta & -\zeta & \chi & \xi \\ \zeta & \eta & -\xi & \chi \end{bmatrix} \tag{9}$$

Quaternion algebra, as opposed to the interpolation method, is used to derive the relative quaternion state transition matrix as a function of $\hat{\omega}_x$, $\hat{\omega}_y$, $\hat{\omega}_z$, Ω_x, Ω_y, Ω_z, and does not require the computation of the direction cosine matrix from the reference space R to the estimate of the spacecraft body fixed space $\hat{B}$. Using this software and assuming perfect gyros, it is found that when the spacecraft is maneuvering at a constant rate the estimate $\hat{q}(t)$ remains as accurate as the postupdate attitude. Gyro errors would thus be the only error source causing $\hat{q}(t)$ to drift from the truth.

Quaternion Algebra

The Euler symmetric parameters ξ, η, ζ, and χ are defined as

$$\xi = E_x \sin(\phi/2),\ \eta = E_y \sin(\phi/2),$$
$$\zeta = E_z \sin(\phi/2),\ \chi = \cos(\phi/2) \tag{10}$$

and represent the rotation of an orthogonal triad R about eigenaxis

$$E = E_x i + E_y j + E_z k \tag{11}$$

through angle ϕ bringing R into coincidence with a second orthogonal triad B. The ordered set $\{i, j, k\}$ defines the R-space orthogonal triad and q becomes

$$q = \chi + i\xi + j\eta + k\zeta \tag{12}$$

where the i, j, k in Eq. (12) are interpreted as hyperimaginary numbers.[1] Suppose we let q represent the rotation of R into B and q' represent the rotation of B into C. The correspondence is

$$q \leftrightarrow [B \leftarrow R]; \qquad q' \leftrightarrow [C \leftarrow B] \tag{13}$$

The quaternion representation of $[C \leftarrow R]$ can be denoted q'' and is given by

$$q'' = qq' \tag{14}$$

Using the rules for quaternion multiplication,[2,3] it can easily be shown that the corresponding Euler symmetric parameters satisfy the equations

$$\begin{bmatrix} \xi'' \\ \eta'' \\ \zeta'' \\ \chi'' \end{bmatrix} = \begin{bmatrix} \chi & -\zeta & \eta & \xi \\ \zeta & \chi & -\xi & \eta \\ -\eta & \xi & \chi & \zeta \\ -\xi & -\eta & -\zeta & \chi \end{bmatrix} \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \\ \chi' \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \xi'' \\ \eta'' \\ \zeta'' \\ \chi'' \end{bmatrix} = \begin{bmatrix} \chi' & \zeta' & -\eta' & \xi' \\ -\zeta' & \chi' & \xi' & \eta' \\ \eta' & -\xi' & \chi' & \zeta' \\ -\xi' & -\eta' & -\zeta' & \chi' \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \chi \end{bmatrix} \tag{16}$$

Letting q denote the quadruple $(\xi, \eta, \zeta, \chi)^T$ of Euler symmetric parameters and similarly for q' and q'', the matrix Eqs. (15) and (16) can be compactly written as

$$q'' = m(q)q' = m^{\dagger}(q')q \tag{17}$$

where $m(q)$ and $m^{\dagger}(q')$ denote the corresponding 4×4 matrices. $m^{\dagger}$ is sometimes called the quaternion transmuted matrix.[4]

State Transition Relation

The state transition relation for Eq. (2) is usually derived by the interpolation method. The matrix elements of $\Delta\Omega$ are assumed constant over the transition interval $[t, t+\Delta t]$. For Eq. (5) with Ω defined by Eq. (6) the state transition relation is derived within the geometric framework of quaternions.

Define four orthogonal triads

S_1 = reference triad at time t
S_2 = reference triad at time $t+\Delta t$
S_3 = spacecraft body fixed triad at time t
S_4 = spacecraft body fixed triad at time $t+\Delta t$

and the following quaternions:

$$q_{4-1} \leftrightarrow [S_4 \leftarrow S_1]$$
$$q_{3-1} \leftrightarrow [S_3 \leftarrow S_1]$$
$$q_{4-3} \leftrightarrow [S_4 \leftarrow S_3]$$
$$q_{2-1} \leftrightarrow [S_2 \leftarrow S_1]$$
$$q_{4-2} \leftrightarrow [S_4 \leftarrow S_2]$$

The quaternion q_{4-1} can be synthesized as

$$q_{4-1} = q_{3-1}q_{4-3} = q_{2-1}q_{4-2} \tag{18}$$

Consequently,

$$q_{4-2} = q^*_{2-1}q_{3-1}q_{4-3} \tag{19}$$

where q^*_{2-1} denotes the conjugate of q_{2-1} which is the rotation of the S_2 into S_1 or q_{1-2}. Letting $q(t)$ denote the quaternion representation of $[S_3 \leftarrow S_1]$, $q(t+\Delta t)$ becomes the quaternion representation of $[S_4 \leftarrow S_2]$. Using Eq. (16), $q_{3-1}\, q_{4-3}$ can be written as

$$m(q_{3-1})q_{4-3} = m^\dagger(q_{4-3})q(t) \tag{20}$$

and using Eq. (15), $q(t+\Delta t)$ can be written as

$$q(t+\Delta t) = m(q_{1-2})m^\dagger(q_{4-3})q(t) \tag{21}$$

The state transition relation becomes

$$\Phi(t+\Delta t,t) = m(q_{1-2})m^\dagger(q_{4-3}) \tag{22}$$

which, assuming constant rates over $[t, t+\Delta t]$, becomes

$$\Phi(t+\Delta t,t) = \left\{\cos\frac{\phi_R}{2} I + \frac{\sin(\phi_R/2)}{\|\Omega\|}\Omega_R\right\}\left\{\cos\frac{\phi_v}{2} I + \frac{\sin(\phi_v/2)}{\|\hat\omega\|}\Omega_v\right\} \tag{23}$$

where

$$\Omega_R = \begin{bmatrix} 0 & \Omega_z & -\Omega_y & -\Omega_x \\ -\Omega_z & 0 & \Omega_x & -\Omega_y \\ \Omega_y & -\Omega_x & 0 & -\Omega_z \\ \Omega_x & \Omega_y & \Omega_z & 0 \end{bmatrix} \tag{24}$$

$$\Omega_v = \begin{bmatrix} 0 & \hat\omega_z & -\hat\omega_y & \hat\omega_x \\ -\hat\omega_z & 0 & \hat\omega_x & \hat\omega_y \\ \hat\omega_y & -\hat\omega_x & 0 & \hat\omega_z \\ -\hat\omega_x & -\hat\omega_y & -\hat\omega_z & 0 \end{bmatrix} \tag{25}$$

with

$\Omega = (\Omega_x, \Omega_y, \Omega_z)^T$ in R space at time t
$\hat\omega = (\hat\omega_x, \hat\omega_y, \hat\omega_z)^T$ in B space at time t
$\phi_R = \|\Omega\|\Delta t$
$\phi_v = \|\hat\omega\|\Delta t$

Note that Ω_v and Ω_R commute so that the factors in Eq. (23) commute. In fact, the interpolation technique can be used as an alternate derivation by first using

$$\exp(\Omega\Delta t/2) = \exp(\Omega_v\Delta t/2)\ \exp(\Omega_R\Delta t/2) \tag{26}$$

Results

Both techniques for propagating the relative quaternion forward in time were studied for the case of a spacecraft spinning at a constant rate. A rate of 2 deg/s about the z axis was chosen. q propagation uses Eqs. (3) and (4) and depends on the body rates, the reference rates, and the transformation from reference space, R space, to the estimate of the spacecraft body fixed space, $\hat B$ space. q_N propagation uses Eq. (23) and depends on the body rates and reference rates only. Both quaternion algorithms were driven with body rate and reference rate profiles. For simplicity, the body rate and reference rate profiles were taken equal. Only the initial quaternion representing the post-update attitude error and the gyro sample period were varied during the study. Three cases were initially studied as defined in Table 1. All initial post-update attitude errors were obtained by rotating the spacecraft body fixed coordinate set, B space, from coincidence with R space to form $\hat B$ space, the onboard estimate of B space. All three cases use an initial attitude error ϵ_x of 10 arc-s. In truth, B space tracks R space and the relative quaternion can be used to extract the angular error

$$\psi = (\psi_x, \psi_y, \psi_z)$$

from the truth. Figure 1 shows the angular error from the truth for the q_N propagation and case 1. The initial 10-arc-s post-update attitude error commutates between the x and y axes. The z-axis error remains zero. The q propagation diverges from the truth as shown in Fig. 2 reaching roughly 20 arc-s in 20 min. The angular error between q and q_N is plotted in Fig. 3 showing a gradual divergence to 10 arc-s in 20 min. The eigenaxis angular error from the truth is shown in Fig. 4 and represents the single axis angular rotation required to bring the B space estimate relative to R space into coincidence with the truth. The q_N propagation maintains the initial post-update attitude error of 10 arc-s whereas the q propagation diverges to almost 21 arc-s.

The results at 20 min for case 2 using a 10-s sample period are summarized in Table 2. The large sample period was

Table 1 Case definition

Case number	Initial attitude error, arc-s ϵ_x	ϵ_y	ϵ_z	Sample period, s	Spin rate, deg/s
1	10	0	0	1	2
2	10	0	0	10	2
3	10	0	0	0.5	2

Table 2 Case 2 summary errors at 20 min, arc-s

	$\lvert\psi_x\rvert_{pk}$	$\lvert\psi_y\rvert_{pk}$	$\lvert\psi_z\rvert_{pk}$	Eigenaxis angular error
q_N propagation	10.	10.	0.	10.
q propagation	8.E3	8.E3	6.E3	9.9E3

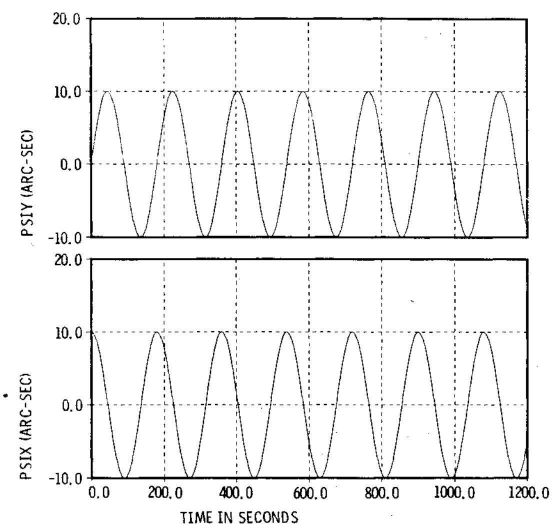

Fig. 1 Angular error from truth for the q_N propagation, case 1.

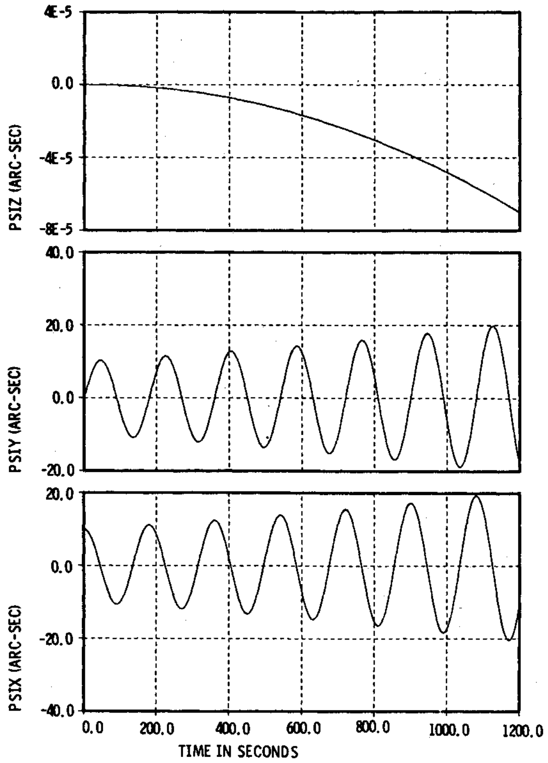

Fig. 2 Angular error from truth for the q propagation, case 2.

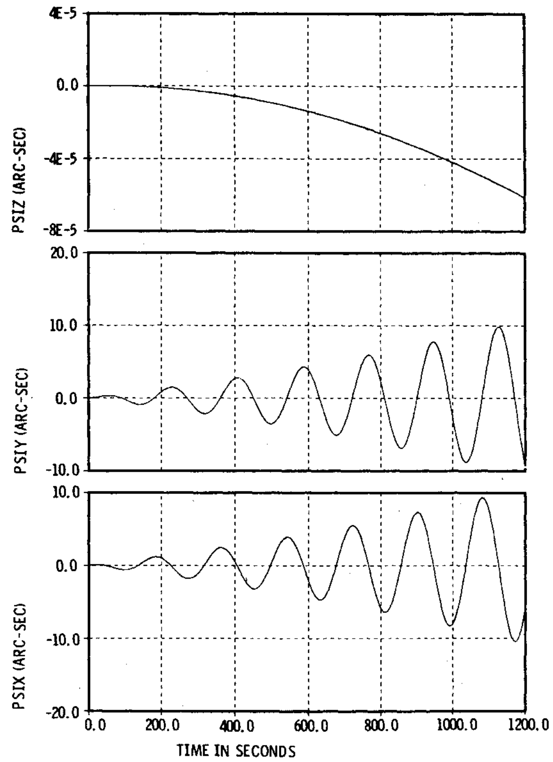

Fig. 3 Angular error between q and q_N, case 1.

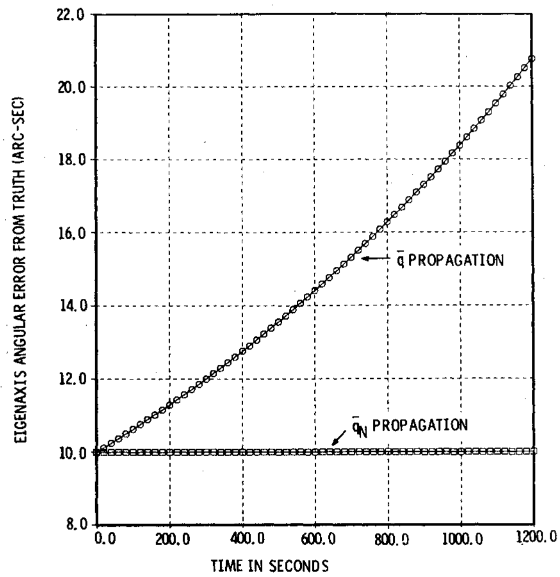

Fig. 4 Eigenaxis angular error from truth, case 1.

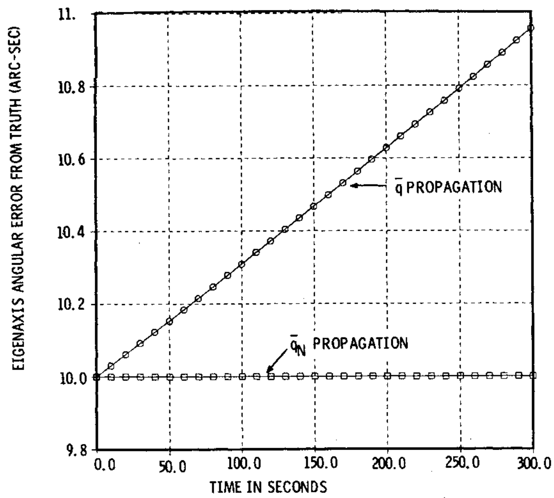

Fig. 5 Eigenaxis angular error from truth, case 3.

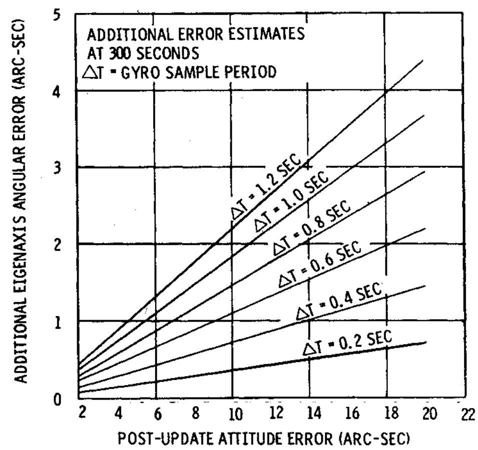

Fig. 6 Linear estimates of the incremental attitude error caused by the q algorithm.

chosen to illustrate the dependence of both algorithms on the sample period. The q_N propagation, however, remained stable with the 10 arc-s initial post-update attitude error commutating between the x and y axes, and the z-axis error remaining zero. The q propagation diverges on all axes with a total single axis angular error or eigenaxis angular error of almost 1.E4 arc-s in 20 min.

The results for case 3 are summarized in Fig. 5, which is a plot of the eigenaxis angular error from the truth using a sample period of 0.5 s and starting with an initial post-update attitude error of 10 arc-s, After 5 min, the error in q is about 10.95 arc-s, while the error in q_N remains 10 arc-s. The incremental angular error is slightly convex upward. Using the tangent constructed at time zero, the incremental angular error can be estimated at 5 min for various sample periods and post-update attitude errors. Because of the convexity in the error, the linear estimate is actually a lower bound on the incremental attitude error caused by the q algorithm. The linear estimates are plotted in Fig. 6 for a satellite spinning at 2 deg/s.

Summary

The state transition relation for propagating the relative quaternion forward in time is derived as a function of the reference space rates in reference space and the estimates of the spacecraft inertial rates in the spacecraft body fixed coordinates. Consequently, the relative quaternion can be propagated forward in time without computing the transformation matrix from the reference space to body coordinates. For a spacecraft spinning at a constant rate this technique for propagating the relative quaternion forward in time remains stable, preserving the initial error in the estimate of the relative attitude of the spacecraft with respect to the noninertial reference space. The standard technique which relies on Eqs. (3) and (4) is unstable causing the relative attitude estimate to diverge at a rate which is a function of the spacecraft spin rate, the post-update attitude error, and the gyro sample period.

The spacecraft body fixed reference triad and the noninertial desired reference can both be tracked relative to the same inertial frame using quaternions. The relative quaternion can thus be synthesized without the use of the relative quaternion state transition relation but at the expense of implementing additional software. For this reason only the stability of the relative quaternion transition relation, as expressed by Eqs. (3) and (4), has been presented and a stable alternate analytic form [see Eq. (23)] derived and recommended.

Acknowledgment

The author wishes to thank J.H. Decanini, D.P. Hoffman, and R.P. Iwens of TRW for their helpful comments and suggestions in the preparation of this manuscript.

References

[1] Mortensen, R.E., "On Systems for Automatic Control of the Rotation of a Rigid Body," University of California, Berkeley, Electronics Research Lab. Rept. 63-23.

[2] Whittaker, E.T., *Analytical Dynamics,* Dover, New York, 1944.

[3] Hendley, A.C., "Quaternions for Control of Space Vehicles," *Proceedings of the ION National Space Meeting of Space Shuttle, Space Station, and Nuclear Shuttle Navigation,* George C. Marshall Space Flight Center, Huntsville, Ala., Feb. 1971.

[4] Ickes, B.P., "A New Method for Performing Digital Control System Attitude Control Computations Using Quaternions," *AIAA Journal,* Vol. 8, Jan. 1970, pp. 13-17.